An alternative complete system of invariants for matrix pencils under strict equivalence

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Abstract:- We consider pairs of matrices (E, A), representing singular linear time invariant systems in the form $E\dot{x}(t) = Ax$ with $E, A \in M_{p\times n}(C)$ under restricted equivalence.

In this paper we obtain an alternative collection of invariants that they permit us to deduce the Kronecker canonical reduced form.

Key-Words:- Singular linear systems, canonical reduced form, structural invariants.

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1 Introduction

We denote by $M_{p\times n}(C)$ the space of complex matrices having p rows and n columns, and in the case which $p = n$ we write $M_n(C)$.

We consider the set $\mathcal{M}$ of pairs of matrices (E, A) representing families of singular linear time invariant systems in the form $E\dot{x} = Ax$ with $E, A \in M_{p\times n}(C)$.

We deal with equivalence relation between systems accepting one or both of the following transformations: basis change in the state space and premultiplication by an invertible manifold. A canonical reduced form can be derived associating a matrix pencil to the system characterized by two sets of minimal indices and sets of finite and infinite elementary divisors (the classical invariants). In this paper we present an alternative complete system of structural invariants based in computation the ranks of certain matrices, which permits an easy characterization of the equivalence classes. An alternative system of structural invariants also based in computation the ranks of certain matrices for standard systems was obtained by García-Planas and Magret in [5]. The Kronecker canonical form provides a variety of applications in control systems theory as we can see in [3], [7] among others.

In the sequel we will use the following notations.

- I_n denotes the n-order identity matrix,

- N denotes a nilpotent matrix in its reduced form $N = \mathrm{diag}(N_1, \ldots, N_\ell)$, $N_i = \begin{pmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{pmatrix} \in M_{n_i}(C)$,

- J denotes the Jordan matrix $J = \mathrm{diag}(J_1, \ldots, J_t)$, $J_i = \mathrm{diag}(J_{i_1}, \ldots, J_{i_s})$, $J_{i_j} = \lambda_i I + N$,

- $L = \mathrm{diag} = (L_1, \ldots, L_q)$, $L_j = \begin{pmatrix} I_{n_j} & 0 \end{pmatrix} \in M_{n_j\times(n_j+1)}(C)$,

- $R = \mathrm{diag}(R_1, \ldots, R_p)$, $R_{n_j} = \begin{pmatrix} 0 & I_{n_j} \end{pmatrix} \in M_{n_j\times(n_j+1)}(C)$.

2 Strict equivalence

The standard transformations in state space: $x(t) = Px_1(t)$, and premultiplication by an invertible matrix: $QE\dot{x}(t) = QAx(t)$, realized over generalized systems relate them in the following manner, two systems are related when one can be obtained from the other by means of one, or more, of the transformations considered. In fact, this transformations define an equivalence relation in the corresponding space of pairs of matrices in the following manner.

Definition 1 *two pairs of matrices $(E_i, A_i) \in \mathcal{M}$, $i = 1, 2$, are equivalent if and only if there exist matrices $P \in Gl(n;\mathbb{C})$ and $Q \in Gl(p;\mathbb{C})$ such that*

$$(E_2, A_2) = \quad (QE_1P, QA_1P),$$

That we can write in a matrix form in the following manner

$$\begin{pmatrix} E_2 & 0 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} E_1 & 0 \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}$$

It is straightforward that the relation defined is an equivalence relation.

3 Associated pencil

Given a pair of matrices $(E, A) \in \mathcal{M}$ we can associate in a natural way the following matrix pencil $H(\lambda) = \lambda E + A$. It is easy to proof the following proposition

Proposition 1 *Two pairs of matrices in $\mathcal{M}$ are equivalent if and only if the associated pencils are strictly equivalent.*

As a consequence we can apply Kronecker's theory of pencils of matrices as presented in [4].

Corollary 1 *Let $(E, A) \in \mathcal{M}$ a pair of matrices. Then, the associated pencil is equivalent to $\lambda F + G$ with*

$$F = \begin{pmatrix} L & & & \\ & L^T & & \\ & & I_1 & \\ & & & N \end{pmatrix} \text{ and } G = \begin{pmatrix} R & & & \\ & R^T & & \\ & & J & \\ & & & I_2 \end{pmatrix}$$

Remember that

Definition 2 *Let $(E, A) \in \mathcal{M}$ be a pair of matrices and $H(\lambda)$ its associated pencil. The value $\lambda_0 \in \mathbb{C}$ is an eigenvalue of $H(\lambda)$ if and only if*

$$\operatorname{rank} H(\lambda_0) < \operatorname{rank} H(\lambda)$$

We denote by $\sigma(E, A)$ the spectrum of the pencil, that is to say the set of eigenvalues of the pencil:

$$\sigma(E, A) = \{\lambda_i \in \mathbb{C} \mid \operatorname{rank} H(\lambda_i) < \operatorname{rank} H(\lambda)\}.$$

It is easy to observe that $\sigma(E, A)$ is an empty set or it is a finite set.

Proposition 2 *The eigenvalues are invariant under equivalence relation considered.*

Theorem 1 *Let $\big(\lambda E + A\big)$ be a matrix pencil under strict equivalence. Each equivalence class is characterized by the following set of structural invariants.*

i) $\omega_1 \geq \cdots \geq \omega_s \geq 1$*: Segre characteristic of infinite zeroes.*

ii) $k_1(\lambda) \geq \cdots \geq k_{j(\lambda)}(\lambda) \geq 1$*: Segre characteristic of eigenvalue λ.*

iii) $\epsilon_1 \geq \cdots \geq \epsilon_{r_\epsilon} > \epsilon_{r_\epsilon+1} = \cdots = \epsilon_r = 0$*: column minimal indices.*

iv) $(\eta_1 \geq \cdots \geq \eta_{l_\eta} > \eta_{l_\eta+1} = \cdots = \eta_l = 0$*: row minimal indices.*

Corollary 2 *Let $\lambda E + A \in M_{p\times n}$ be a matrix pencil. Then*

$$n = \sum_{i=1}^{s} \omega_i + \sum_{i=1}^{u} \sum_{j=1}^{j(\lambda_i)} k_j(\lambda_i) + \sum_{i=1}^{l_\eta} \eta_i + \sum_{i=1}^{r_\epsilon} \epsilon_i + r_\epsilon + r_0.$$

$$p = \sum_{i=1}^{s} \omega_i + \sum_{i=1}^{u} \sum_{j=1}^{j(\lambda_i)} k_j(\lambda_i) + \sum_{i=1}^{r_\epsilon} \epsilon_i + \sum_{i=1}^{l_\eta} \eta_i + l_\eta + l_0.$$

where $r_0 = r - r_\epsilon$ and $l_0 = l - l_\eta$ denote the number of zero columns and zero rows respectively.

Definition 3 *Given a pair of matrices (E, A), we call rank of the pair and we will denote by r_n to the rank of the associated pencil $H(\lambda)$:*

$$r_n = \operatorname{rank}(E, A) = \operatorname{rank}(\lambda E + A).$$

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3 Sequences of matrices associated to (E, A)

Let $(E, A) \in \mathcal{M}$ be a pair of matrices, for all $\ell = 1, 2, \dots$ we define the matrices

$\mathcal{H}_0 = E$,

$\mathcal{H}_1 = \begin{pmatrix} E & \\ A & E \end{pmatrix}$,

$$\mathcal{H}_\ell = \begin{pmatrix} E & & & \\ A & E & & \\ & A & \ddots & \\ & & \ddots & \\ & & & E \end{pmatrix},$$

$\mathcal{C}_1 = \begin{pmatrix} E \\ A \end{pmatrix}$,

$$\mathcal{C}_\ell = \begin{pmatrix} E & & & \\ A & E & & \\ & A & \ddots & \\ & & \ddots & E \\ & & & A \end{pmatrix},$$

$\mathcal{O}_1 = \begin{pmatrix} E & A \end{pmatrix}$,

$$\mathcal{O}_\ell = \begin{pmatrix} E & A & & & \\ & E & A & & \\ & & \ddots & \ddots & \\ & & & E & A \end{pmatrix},$$

$\mathcal{J}_0 = \lambda E + A$,

$\mathcal{J}_1(\lambda) = \begin{pmatrix} \lambda E + A & \\ E & \lambda E + A \end{pmatrix}$,

$$\mathcal{J}_\ell(\lambda) = \begin{pmatrix} \lambda E + A & & & \\ E & \lambda E + A & & \\ & E & \ddots & \\ & & \ddots & \\ & & & \lambda E + A \end{pmatrix}, \ \lambda \in \mathbb{C}.$$

Proposition 3 *The ranks of the matrices $\mathcal{H}_\ell$, $\mathcal{J}_\ell(\lambda)$, $\mathcal{C}_\ell$, $\mathcal{O}_\ell$, for all $\ell = 1, 2, \dots$ are invariant under equivalence relation considered.*

We denote by

- $r^{\mathcal{H}}_\ell = \operatorname{rank} \mathcal{H}_\ell$,
- $r_\ell(\lambda) = \operatorname{rank} \mathcal{J}_\ell(\lambda)$,
- $r^{\mathcal{C}}_\ell = \operatorname{rank} \mathcal{C}_\ell$ and
- $r^{\mathcal{O}}_\ell = \operatorname{rank} \mathcal{O}_\ell$.

Theorem 2 ([6]) *Let $(E, A) \in \mathcal{M}$ be a pair of matrices of rank r_n. Then, for all $\ell = 1, 2, \dots$*

$$\begin{aligned} r^{\mathcal{H}}_\ell &= r_n \ell - \textstyle\sum_{i=1}^{s} \min\{\ell, \omega_i\} \\ r_\ell(\lambda) &= r_n \ell - \textstyle\sum_{i=1}^{j(\lambda)} \min\{\ell, k_i(\lambda)\} \\ r^{\mathcal{C}}_\ell &= r_n \ell + \textstyle\sum_{i=1}^{r_\epsilon} \min\{\ell, \epsilon_i\} \\ r^{\mathcal{O}}_\ell &= r_n \ell + \textstyle\sum_{i=1}^{l_\eta} \min\{\ell, \eta_i\} \end{aligned}$$

Proof. It suffices to compute these collection of numbers, for an equivalent pair such that the associated pencil is in its Kronecker reduced form. □

It is easy to deduce the following corollaries.

Corollary 3 $s = r_n - r^{\mathcal{H}}_1$, $j(\lambda) = r_n - r_1(\lambda)$, $r_\epsilon = r^{\mathcal{C}}_1 - r_n$, $l_\eta = r^{\mathcal{O}}_1 - r_n$ *and* $r_n = r^{\mathcal{H}}_{n+1} - r^{\mathcal{H}}_n = r^{\mathcal{C}}_{n+1} - r^{\mathcal{C}}_n = r^{\mathcal{O}}_{n+1} - r^{\mathcal{O}}_n = r_{n+1}(\lambda) - r_n(\lambda)$.

Corollary 4 *For all pair $(E, A) \in \mathcal{M}$ we have*

$$\begin{aligned} r_n &= r^{\mathcal{H}}_{n+1} - r^{\mathcal{H}}_n = \\ &= \textstyle\sum_{i=1}^{s} \omega_i + \sum_{i=1}^{u} \sum_{j=1}^{j(\lambda_i)} k_j(\lambda_i) + \\ &+ \textstyle\sum_{i=1}^{r_\epsilon} \epsilon_i + \sum_{i=1}^{l_\eta} \eta_i. \end{aligned}$$

Theorem 3 *Let $(E, A) \in \mathcal{M}$ a pair of matrices. Then*

1. *The numbers*

$$\begin{aligned} s_1 &= r^{\mathcal{H}}_1 - 2r^{\mathcal{H}}_0 \\ s_{\ell+1} &= r^{\mathcal{H}}_{\ell+1} - 2r^{\mathcal{H}}_\ell + r^{\mathcal{H}}_{\ell-1}, \ \ell = 1, 2, \dots \end{aligned}$$

determine the quantity of blocks of size ℓ corresponding to the infinite zeroes in the Kronecker reduced form of the associated pencil.

2. *The numbers*

$$\begin{aligned} j_1(\lambda) &= r_1(\lambda) - 2r_0(\lambda) \\ j_{\ell+1}(\lambda) &= r_{\ell+1}(\lambda) - 2r_\ell(\lambda) + r_{\ell-1}(\lambda), \\ &\quad \ell = 1, 2, \dots \end{aligned}$$

determine the quantity of Jordan blocks of size ℓ associated to the eigenvalue λ of the pencil $H(\lambda)$.

Proof. It suffices to compute the ranks of the matrices $\mathcal{H}_i$ of a Kronecker reduced form pencil. □

Remark 2 let $(E, A) \in \mathcal{M}$ a pair of matrices and $H(\lambda)$ its associated pencil. If λ is not in the spectrum of $H(\lambda)$, we have that $j_\ell(\lambda) = 0, \ \ell = 1, 2, \dots$.

Theorem 4 ([9]) *Let $(E, A) \in \mathcal{M}$ be a pair of matrices. Then,*

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1. *the numbers*
   $$r_1 = -r_2^{\mathcal{C}} + 2r_1^{\mathcal{C}}$$
   $$r_\ell = -r_{\ell+1}^{\mathcal{C}} + 2r_\ell^{\mathcal{C}} - r_{\ell-1}^{\mathcal{C}},\ \ell = 2, 3, \dots$$
   determine the quantity of column minimal indices of size ℓ appearing in $\lambda E + A$,

2. *the numbers*
   $$l_1 = -r_2^{\mathcal{O}} + 2r_1^{\mathcal{O}}$$
   $$l_\ell = -r_{\ell+1}^{\mathcal{O}} + 2r_\ell^{\mathcal{O}} - r_{\ell-1}^{\mathcal{O}},\ \ell = 2, 3, \dots$$
   determine the quantity of row minimal indices of size ℓ appearing in $\lambda E + A$.

5 An alternative complete systems of invariants

Now we construct an alternative method to obtain the canonical reduced form of a pencil $\lambda E + A$. These method is deduced from the following numbers $r_\ell^{\mathcal{H}}$, $r_\ell(\lambda)$, $r_\ell^{\mathcal{C}}$ y $r_\ell^{\mathcal{O}}$, $\ell = 1, 2, \dots$.

Definition 4 *For all pair of matrices (E, A), we will call infinite zeros numbers and we will write $r_i^{\mathcal{CO}}$, to*

$$\begin{aligned} r_1^{\mathcal{CO}} &= r_n - r_1^{\mathcal{H}} \\ r_2^{\mathcal{CO}} &= r_1^{\mathcal{H}} - r_2^{\mathcal{H}} + r_n \\ &\vdots \\ r_\ell^{\mathcal{CO}} &= r_{\ell-1}^{\mathcal{H}} - r_\ell^{\mathcal{H}} + r_n, \quad \ell = 2, 3, \dots \end{aligned}$$

Proposition 4 *Let $(E, A) \in \mathcal{M}$ be a pair and $H(\lambda)$ its associated pencil. Each $r_\ell^{\mathcal{CO}}, \ell = 1, 2, \dots$ determine the quantity of blocks corresponding to the infinite zeroes of size greater than $\ell - 1$, that they appear in $H(\lambda)$. The $r^{\mathcal{CO}}$-numbers verify:*

$$r_1^{\mathcal{CO}} \geq r_2^{\mathcal{CO}} \geq \cdots \geq r_{\ell_1}^{\mathcal{CO}} \geq r_{\ell_1+1}^{\mathcal{CO}} = \cdots = 0.$$

Proof.

$$\begin{aligned} s =&\ r_n - r_1^{\mathcal{H}} \\ s - s_1 =&\ r_1^{\mathcal{H}} - r_2^{\mathcal{H}} + r_n \\ &\vdots \\ s - \textstyle\sum_{i=1}^{\ell-1} s_i =&\ r_{\ell-1}^{\mathcal{H}} - r_\ell^{\mathcal{H}} + r_n, \\ \ell = 2, 3, \dots& \end{aligned}$$

□

Corollary 5 *The infinite zeroes indices $(\omega_1, \omega_2, \dots, \omega_s)$ are the conjugate partition of the set of non zero numbers $r_\ell^{\mathcal{CO}}$.*

Definition 5 *For all pair of matrices (E, A), we will call Jordan numbers corresponding to the eigenvalue λ and we will write $r_i^{\overline{\mathcal{CO}}}(\lambda)$, to*

$$\begin{aligned} r_1^{\overline{\mathcal{CO}}}(\lambda) &= r_n - r_1(\lambda) \\ r_2^{\overline{\mathcal{CO}}}(\lambda) &= r_1(\lambda) - r_2(\lambda) + r_n \\ &\vdots \\ r_\ell^{\overline{\mathcal{CO}}}(\lambda) &= r_{\ell-1}(\lambda) - r_\ell(\lambda) + r_n, \\ \ell =&\ 2, 3, \dots \end{aligned}$$

Proposition 5 *Let $(E, A) \in \mathcal{M}$ be a pair and $H(\lambda)$ its associated pencil. Each $r_\ell^{\overline{\mathcal{CO}}}(\lambda), \ell = 1, 2, \dots$ determine the quantity of Jordan blocks of size greater than $\ell - 1$ associated to the eigenvalue λ appearing in $\lambda E + A$. The $r^{\overline{\mathcal{CO}}}$-numbers verify:*

$$\begin{aligned} r_1^{\overline{\mathcal{CO}}}(\lambda) \geq \cdots \geq r_{\ell(\lambda)}^{\overline{\mathcal{CO}}}(\lambda) \geq \\ \geq r_{\ell(\lambda)+1}^{\overline{\mathcal{CO}}}(\lambda) = \cdots = 0. \end{aligned}$$

Proof.

$$\begin{aligned} j(\lambda) =&\ r_n - r_1(\lambda) \\ j(\lambda) - j_1(\lambda) =&\ r_1(\lambda) - r_2(\lambda) + r_n \\ &\vdots \\ j(\lambda) - \textstyle\sum_{i=1}^{\ell-1} j_i(\lambda) =&\ r_{\ell-1}(\lambda) - r_\ell(\lambda) + r_n, \\ \ell = 2, 3, \dots.& \end{aligned}$$

□

Corollary 6 *The Segre characteristic $(k_1(\lambda), k_2(\lambda), \dots, k_{j(\lambda)}(\lambda))$ is the conjugate partition of non zero $r_\ell^{\overline{\mathcal{CO}}}(\lambda)$ numbers.*

Definition 6 *For all pair of matrices (E, A), we will call column minimal numbers and we will write $r_i^{\mathcal{C}\overline{\mathcal{O}}}$ to*

$$\begin{aligned} r_0^{\mathcal{C}\overline{\mathcal{O}}} &= n - r_n \\ r_1^{\mathcal{C}\overline{\mathcal{O}}} &= r_1^{\mathcal{C}} - r_n \\ r_2^{\mathcal{C}\overline{\mathcal{O}}} &= r_2^{\mathcal{C}} - r_1^{\mathcal{C}} - r_n \\ &\vdots \\ r_\ell^{\mathcal{C}\overline{\mathcal{O}}} &= r_\ell^{\mathcal{C}} - r_{\ell-1}^{\mathcal{C}} - r_n, \quad \ell = 2, 3, \dots \end{aligned}$$

Proposition 6 *Let $(E, A) \in \mathcal{M}$ be a pair of matrices and $H(\lambda)$ its associated pencil each $r_i^{\mathcal{C}\overline{\mathcal{O}}}$ number determine the quantity of column minimal indices of size greater than $\ell - 1$ that appear in $\lambda E + A$. These $r_i^{\mathcal{C}\overline{\mathcal{O}}}$-numbers verify*

$$r_0^{\mathcal{C}\overline{\mathcal{O}}} \geq r_1^{\mathcal{C}\overline{\mathcal{O}}} \geq \cdots \geq r_{\ell_2-1}^{\mathcal{C}\overline{\mathcal{O}}} \geq r_{\ell_2}^{\mathcal{C}\overline{\mathcal{O}}} = \cdots = 0$$

Proof.

$$\begin{aligned} r = r_\epsilon + r_0 = &\ n - r_n \\ r_\epsilon = &\ r_1^{\mathcal{C}} - r_n \\ r_\epsilon - r_1 = &\ r_2^{\mathcal{C}} - r_1^{\mathcal{C}} - r_n \\ &\vdots \\ r_\epsilon - \textstyle\sum_{i=1}^{\ell-1} r_i = &\ r_\ell^{\mathcal{C}} - r_{\ell-1}^{\mathcal{C}} - r_n,\ \ell = 2, 3, \ldots \end{aligned}$$

□

Corollary 7 *let $(k_1^\epsilon, \ldots, k_{r_\epsilon}^\epsilon, k_{r_\epsilon+1}^\epsilon, \ldots, k_r^\epsilon)$ be the conjugate partition of the non zero $r_i^{\mathcal{C}\overline{\mathcal{O}}}$-numbers. Then the non-negative numbers*

$$\begin{aligned} &(\epsilon_1, \ldots, \epsilon_{r_\epsilon}, 0, \ldots, 0) = \\ &(k_1^\epsilon - 1, \ldots, k_{r_\epsilon}^\epsilon - 1, k_{r_\epsilon+1}^\epsilon - 1, \ldots, k_r^\epsilon - 1) \end{aligned}$$

coincide with the column minimal indices that appear in $\lambda E + A$.

As a consequence we have.

Corollary 8 *$\lambda E + A$ has column full rank if and only if $r_n = n$.*

Corollary 9 *Let $\lambda E + A$ be a pencil having column full rank. Then, for each ℓ, matrices $\mathcal{C}_\ell(E, A)$ have column full rank.*

Proof.

$$r_1^{\mathcal{C}\overline{\mathcal{O}}} + \cdots + r_\ell^{\mathcal{C}\overline{\mathcal{O}}} = r_\ell^{\mathcal{C}} - \ell r_n = r_\ell^{\mathcal{C}} - \ell n = 0.$$

□

Definition 7 *For all pair of matrices (E, A), we will call row minimal numbers and we will write $r_i^{\overline{\mathcal{C}}\mathcal{O}}$ to*

$$\begin{aligned} r_0^{\overline{\mathcal{C}}\mathcal{O}} &= p - r_n \\ r_1^{\overline{\mathcal{C}}\mathcal{O}} &= r_1^{\mathcal{O}} - r_n \\ r_2^{\overline{\mathcal{C}}\mathcal{O}} &= r_2^{\mathcal{O}} - r_1^{\mathcal{O}} - r_n \\ &\vdots \\ r_\ell^{\overline{\mathcal{C}}\mathcal{O}} &= r_\ell^{\mathcal{O}} - r_{\ell-1}^{\mathcal{O}} - r_n, \quad \ell = 2, 3, \ldots \end{aligned}$$

Proposition 7 *Let $(E, A) \in \mathcal{M}$ be a pair of matrices and $H(\lambda)$ its associated pencil each $r_i^{\overline{\mathcal{C}}\mathcal{O}}$ number determine the quantity of row minimal indices of size greater than $\ell - 1$ that appear in $\lambda E + A$. These $r_i^{\overline{\mathcal{C}}\mathcal{O}}$-numbers verify*

$$r_0^{\overline{\mathcal{C}}\mathcal{O}} \geq \cdots \geq r_{\ell_3-1}^{\overline{\mathcal{C}}\mathcal{O}} \geq r_{\ell_3}^{\overline{\mathcal{C}}\mathcal{O}} = \cdots = 0$$

Proof.

$$\begin{aligned} l = l_\eta + l_0 = &\ p - r_n \\ l_\eta = &\ r_1^{\mathcal{O}} - r_n \\ l_\eta - l_1 = &\ r_2^{\mathcal{O}} - r_1^{\mathcal{O}} - r_n \\ &\vdots \\ l_\eta - \textstyle\sum_{i=1}^{\ell-1} = &\ r_\ell^{\mathcal{O}} - r_{\ell-1}^{\mathcal{O}} - r_n, \ell = 2, 3, \ldots \end{aligned}$$

□

Corollary 10 *Let $(k_1^\eta, \ldots, k_{l_\eta}^\eta, k_{l_\eta+1}^\eta, \ldots, k_l^\eta)$ be the conjugate partition of the non-zero $r_i^{\overline{\mathcal{C}}\mathcal{O}}$ numbers. Then*

$$\begin{aligned} &(\eta_1, \ldots, \eta_{l_\eta}, 0, \ldots, 0) = \\ &(k_1^\eta - 1, \ldots, k_{l_\eta}^\eta - 1, k_{l_\eta+1}^\eta - 1, \ldots, k_l^\eta - 1) \end{aligned}$$

coincide with the row minimal indices that its appear in $\lambda E + A$.

Corollary 11 *$\lambda E + A$ has full row rank if and only if $r_n = p$.*

Corollary 12 *Let $\lambda E + A$ a pencil having full row rank. Then for all ℓ, the matrices $\mathcal{O}_\ell(E, A)$ have full row rank.*

Proof.

$$r_1^{\overline{\mathcal{C}}\mathcal{O}} + \cdots + r_\ell^{\overline{\mathcal{C}}\mathcal{O}} = r_\ell^{\mathcal{O}} - \ell r_n = r_\ell^{\mathcal{O}} - \ell p = 0$$

□

As a consequence we have the following results.

Theorem 7 *The pencil $H(\lambda) = \lambda E + A$ has full column and row rank if and only if $p = n$ and $r_n = n$.*

Theorem 8 *For all pair of matrices $(E, A) \in \mathcal{M}$, the collection of numbers*

i) $r_1^{\mathcal{C}\mathcal{O}} \geq \cdots \geq r_{\ell_1}^{\mathcal{C}\mathcal{O}} \geq r_{\ell_1+1}^{\mathcal{C}\mathcal{O}} = \cdots = 0,$

ii) $r_0^{\mathcal{C}\overline{\mathcal{O}}} \geq \cdots \geq r_{\ell_2-1}^{\mathcal{C}\overline{\mathcal{O}}} \geq r_{\ell_2}^{\mathcal{C}\overline{\mathcal{O}}} = \cdots = 0,$

iii) $r_0^{\bar{\mathcal{C}}\mathcal{O}} \geq \dots \geq r_{\ell_3-1}^{\bar{\mathcal{C}}\mathcal{O}} \geq r_{\ell_3}^{\bar{\mathcal{C}}\mathcal{O}} = \dots = 0,$

iv) $r_1^{\overline{\mathcal{C}\mathcal{O}}}(\lambda) \geq \dots \geq r_{\ell(\lambda)}^{\overline{\mathcal{C}\mathcal{O}}}(\lambda) \geq r_{\ell(\lambda)+1}^{\overline{\mathcal{C}\mathcal{O}}}(\lambda) = \dots = 0, \quad \lambda \in \mathbb{C}$

constitute a complete system of invariants.

Proof. The non-zero r-numbers permit us to deduce the collection of numbers

i) $\omega_1 \geq \dots \geq \omega_s \geq 1$

ii) $k_1(\lambda) \geq \dots \geq k_{j(\lambda)}(\lambda) \geq 1, \quad \lambda \in \sigma(E, A)$

iii) $\epsilon_1 \geq \dots \geq \epsilon_{r_\epsilon} > \epsilon_{r_\epsilon+1} = \dots = \epsilon_r = 0$

iv) $\eta_1 \geq \dots \eta_{l_\eta} > \eta_{l_\eta+1} = \dots = \eta_l = 0$

that correspond with the structural invariants of the associated pencil to the pair of matrices. □

6 Conclusion

We consider pairs of matrices (E, A), representing singular linear time invariant systems in the form $E\dot{x}(t) = Ax(t)$ with $E, A \in M_{p\times n}(C)$ under equivalence that accept basis change in the state space and premultiplication by an invertible matrix. After to observe that this equivalence corresponds with strict equivalence defined over associated pencil $\lambda E+A$ and the Kronecker reduced form can be used, in this paper we obtain an alternative collection of invariants that they permit us to deduce the canonical reduced form.

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